Published by Institute of Physics Publishing for SISSA/ISAS

Received: *December 9, 2004*
Accepted: *January 3, 2005*

Symmetries of generalized soliton models and submodels on target space S^2

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Abstract: Some physically relevant non-linear models with solitons, which have target space S^2, are known to have submodels with infinitely many conservation laws defined by the eikonal equation. Here we calculate all the symmetries of these models and their submodels by the prolongation method. We find that for some models, like the Baby Skyrme model, the submodels have additional symmetries, whereas for others, like the Faddeev-Niemi model, they do not.

Keywords: Solitons Monopoles and Instantons, Space-Time Symmetries, Global Symmetries.

JHEP01(2005)004

Contents

1. Introduction

Non-linear field theories which support extended, soliton type solutions are of importance in various fields of physics, ranging from elementary particle theory to condensed matter. If the fields of the theory are required to approach a constant value at spatial infinity in order to guarantee finite action or energy, then space time $\mathbb{R}\times\mathbb{R}^d$ is topologically equivalent to $\mathbb{R}\times S^d$ (here d is the dimension of space). Static, soliton-like solutions are then maps $S^d \to \mathcal{N}$ (where $\mathcal{N}$ is the target space) and may sometimes be characterized by a topological index. It frequently happens that the energy of a field configuration can be bounded from below by that topological index, which then implies the existence of non-trivial soliton solutions for a non-trivial topological index. If the target space is a sphere S^n then the maps are characterized by the elements of the corresponding homotopy group $\pi_d(S^n)$. Here we shall consider the target space S^2 and the space dimensions $d = 2, 3$ with homotopy groups $\pi_2(S^2) = \mathbb{Z}$ and $\pi_3(S^2) = \mathbb{Z}$, respectively.

Among theories with target space S^2, a well-known theory in $2+1$ dimensional space time is the $CP(1)$ or Baby Skyrme model with lagrangian density

$$\mathcal{L}_2 = \frac{\partial_\mu u\, \partial^\mu \bar{u}}{(1+u\bar{u})^2} \tag{1.1}$$

where u is a complex field, which parametrizes the the stereographic projection of the target S^2 $\mathbf{n} : \mathbb{R}\times\mathbb{R}^d \to S^2$, $\mathbf{n}^2 = 1$, via

$$\mathbf{n} = \frac{1}{1+u\bar{u}}\,(u+\bar{u}, -i(u-\bar{u}), u\bar{u}-1)\ ; \qquad u = \frac{n_1 + i n_2}{1-n_3}\,. \tag{1.2}$$

We use greek indices for space-time components, $\mu, \nu = 0, 1, 2$ or $\mu, \nu = 0, 1, 2, 3$ and latin indices for space components, $j, k = 1, 2$ or $j, k = 1, 2, 3$, respectively.

The probably best-known theory with target space S^2 in $3+1$ dimensions is the Faddeev-Niemi model ([1, 2]) with lagrangian density

$$\mathcal{L}_{\mathrm{FN}} = \mathcal{L}_2 - \lambda \mathcal{L}_4 \tag{1.3}$$

where λ is a dimensionful coupling constant, $\mathcal{L}_2$ is like in eq. (1.1) (but with $\mu = 0, \ldots, 3$) and $\mathcal{L}_4$ is

$$\mathcal{L}_4 = \frac{(\partial^\mu u\, \partial_\mu \bar{u})^2 - (\partial^\mu u\, \partial_\mu u)(\partial^\nu \bar{u}\, \partial_\nu \bar{u})}{(1 + u\bar{u})^4} \,. \tag{1.4}$$

The Faddeev-Niemi model is the S^2 restriction of Skyrme theory and so circumvents Derrick's theorem, because it consists of two terms such that their corresponding energies behave oppositely under a scale transformation. The existence of (static) soliton solutions for the lowest Hopf indices has been confirmed by numerical calculations ([3]–[6]).

Further models with solitons may be constructed from the two lagrangian densities $\mathcal{L}_2$ and $\mathcal{L}_4$ separately by choosing appropriate (non-integer) powers of these lagrangians such that the corresponding energies are scale invariant.[1] For $\mathcal{L}_4$ the appropriate choice is

$$\mathcal{L}_{\mathrm{AFZ}} = -(\mathcal{L}_4)^{\frac{3}{4}} \tag{1.5}$$

and for this model infinitely many analytic soliton solutions were found by Aratyn, Ferreira and Zimerman (=AFZ) by using an ansatz with toroidal coordinates ([8, 9]). We shall, therefore, refer to this model as the AFZ model in the sequel. The analysis of the AFZ model was carried further in ([10]), where, among other results, all the space-time and (geometric) target space symmetries of the AFZ model were determined.

The appropriate choice for $\mathcal{L}_2$ is

$$\mathcal{L}_{\mathrm{Ni}} = (\mathcal{L}_2)^{\frac{3}{2}} \,. \tag{1.6}$$

This model has first been proposed by Nicole ([11]), and it was shown in the same paper that the simplest Hopf map with Hopf index 1 is a soliton solution for this model. To the best of our knowledge, there are no more results on this model available in the literature. We shall refer to this model as the Nicole (=Ni) model in the sequel.

All four models (Baby Skyrme, Faddeev-Niemi, AFZ and Nicole) have the same target space S^2 described by the variable u, therefore they have some common properties. For instance, all lagrangians are invariant under modular transformations

$$u \;\rightarrow\; \frac{au + b}{-\bar{b}u + \bar{a}} \,, \qquad a\bar{a} + b\bar{b} = 1 \,. \tag{1.7}$$

This is a simple consequence of the fact that all four lagrangians are scalars when expressed in terms of the vector $\mathbf{n}$ and are, therefore, invariant under SO(3) rotations of this vector.

[1] Non-polynomial lagrangian densities of this type were first introduced by [7] as possible effective chiral pion models.

Furthermore, the same area-preserving diffeomorphisms on the target space S^2 can be defined for all models, but this does not imply that they are symmetries for all four field theories. In fact, only the AFZ model has the area-preserving diffeomorphisms as symmetries, which may be understood from the fact that the lagrangian density of the AFZ model is just (a power of) the pullback of the area two-form[2] on S^2 under the map u ([10, 12]). For the other three models the generators Q^G of the area-preserving diffeomorphisms (to be defined below) do not generate symmetries and the corresponding Noether currents J^G_μ are not conserved. However, as it was realized within the generalization of the zero curvature representation, [14], there exist submodels for all three theories such that these currents are conserved. The submodels are defined by a further condition (in addition to the equation of motion), which is the same for all three models (up to dimensionality), namely the complex eikonal equation

$$\partial^\mu u \partial_\mu u = 0\,. \tag{1.8}$$

For fields u which obey the equation of motion of the Baby Skyrme, Faddeev-Niemi or Nicole model and, in addition, the complex eikonal equation (1.8), the currents J^G_μ (to be defined below for each model) are conserved for an arbitrary real function G which depends on both u and $\bar{u}$ (but not on derivatives thereof), therefore these submodels have infinitely many conserved charges.

At this point a word of caution is appropriate: the existence of the infinitely many conserved charges does *not* imply that these submodels have the area-preserving diffeomorphisms as symmetries. The crucial issue is that the complex eikonal equation is not of the Euler-Lagrange type, i.e., it does not result from an action principle. Therefore, there is no direct relation between symmetries and conservation laws, and the issue of symmetry has to be investigated separately for the submodels. In any case, the existence of infinitely many conserved charges is nevertheless quite restrictive and might simplify the analysis of these models, which is why we call them integrable, in analogy to the situation in lower dimensions.

It is the purpose of this paper to investigate the symmetries of the Baby Skyrme, Faddeev-Niemi and Nicole model and, especially, of their integrable submodels. Specifically, we study the symmetries of the equations of motion (and of the eikonal equation) for static, time-independent fields u, for simplicity and as they provide the soliton solutions.[3] In section 2 we briefly review the issue of area-preserving diffeomorphisms and of their infinitesimal generators, because we need them in the sequel. In section 3 we study the symmetries of the static complex eikonal equation both in 2 and 3 dimensions, which will be relevant for the integrable submodels. In section 4 we study the symmetries of the static Baby Skyrme model and of its integrable submodel. In section 5 we do the same for the Nicole model, and in section 6 for the Faddeev-Niemi model. Section 7 contains our conclusions. For the symmetry aspects of the AFZ model we refer to ([10, 12, 13]).

[2]So the equations of motion are quadratic in time.

[3]Time evolution of solutions, included in principle in the approach of [14], can be most interesting [13].

2. Area-preserving diffeomorphisms on S^2

An area-preserving diffeomorphism on target space is a transformation $u \to v(u,\bar{u})$ such that the area form on S^2 remains invariant (see also refs. [10] and [12]),

$$\Omega \equiv \frac{1}{2i}\frac{du d\bar{u}}{(1+u\bar{u})^2} = \frac{1}{2i}\frac{dv d\bar{v}}{(1+v\bar{v})^2} \,. \tag{2.1}$$

For infinitesimal transformations $v = u + \epsilon$ it is easy to see that the condition of invariance of the area form leads to

$$\epsilon_u + (\epsilon_u)^* = 2\frac{\bar{u}\epsilon + u\bar{\epsilon}}{1+u\bar{u}} \,. \tag{2.2}$$

Here subscripts mean partial derivatives, $\epsilon_u \equiv \partial_u \epsilon$. Further, we use overbars for the complex conjugate of a variable, but stars to denote the operation of complex conjugation, e.g., $(u)^* = \bar{u}$, $(f_u)^* = \bar{f}_{\bar{u}}$, etc. Defining

$$\epsilon = (1+u\bar{u})^2 f \,, \qquad f = F_{\bar{u}} \tag{2.3}$$

the above equation for ϵ simplifies to

$$\partial_u \partial_{\bar{u}}(F + \bar{F}) = 0 \tag{2.4}$$

which is solved by any purely imaginary function F of u and $\bar{u}$.

(Remark: it seems that the most general solution of (2.4) is any F such that $F + \bar{F} = g(u) + (g(u))^*$; however, such a g which depends on u only may always be reabsorbed by redefining $F \to F + g$ without changing f or ϵ, therefore an arbitrary imaginary F is the most general solution.)

Introducing the real function G via $F = iG$, the area-preserving diffeomorphisms are therefore generated by the vector fields

$$v^G = i(1+u\bar{u})^2 (G_{\bar{u}}\partial_u - G_u \partial_{\bar{u}}) \tag{2.5}$$

which obey the Lie algebra

$$[v^{G_1}, v^{G_2}] = v^{G_3} \,, \qquad G_3 = i(1+u\bar{u})^2 (G_{1,\bar{u}}G_{2,u} - G_{1,u}G_{2,\bar{u}}) \,. \tag{2.6}$$

For field theories with the two-sphere as target space the generators of area-preserving diffeomorphisms may be constructed from the canonical momenta π, $\bar{\pi}$ of the fields u and $\bar{u}$. They read

$$Q^G = i \int d^d\mathbf{x} (1+u\bar{u})^2 (\bar{\pi} G_u - \pi G_{\bar{u}}) \tag{2.7}$$

and act on functions of $u, \bar{u}, \pi, \bar{\pi}$ via the Poisson bracket, where the fundamental Poisson bracket is (with $x^0 = y^0$)

$$\{u(\mathbf{x}), \pi(\mathbf{y})\} = \{\bar{u}(\mathbf{x}), \bar{\pi}(\mathbf{y})\} = \delta^d(\mathbf{x} - \mathbf{y}) \tag{2.8}$$

as usual. The generators Q^{G_i} close under the Poisson bracket, $\{Q^{G_1}, Q^{G_2}\} = Q^{G_3}$ where G_3 is as in (2.6).

Via the Noether charges (generators of area-preserving diffeomorphisms) Q^G the action of infinitesimal area-preserving diffeomorphisms is defined for all four field theories given in section 1.

3. Symmetries of the eikonal equation

We shall use the method of prolongations for all our symmetry calculations, and we shall use the symmetry-generating vector fields in evolutionary form. That is to say, when a vector field $v = \delta u \partial_u + \delta \bar{u} \partial_{\bar{u}}$ acting on the target space variables is given, then the coefficients for the prolongation of the vector field (giving the action on derivatives of the field) are defined as total derivatives of the original coefficient. Concretely, our equations are at most second order, therefore we need the second prolongation of v,

$$\mathrm{pr}^{(2)} v = \delta u \partial_u + \delta \bar{u} \partial_{\bar{u}} + \delta u^j \partial_{u_j} + \delta \bar{u}^j \partial_{\bar{u}_j} + \delta u^{jk} \partial_{u_{jk}} + \delta \bar{u}^{jk} \partial_{\bar{u}_{jk}} \tag{3.1}$$

where $u_j \equiv \partial_{x^j} u$, etc. and the Einstein summation convention is understood. As said, the coefficients for the prolongations are given by total derivatives,

$$\delta u = \phi \quad \Rightarrow \quad \delta u^j = D_j \phi \,, \qquad \delta u^{jk} = D_j D_k \phi \,. \tag{3.2}$$

The explicit expressions for these total derivatives depend on which dependencies are assumed for the function ϕ. Here we assume that ϕ may depend on the independent and dependent variables as well as on the first derivatives of the latter, that is

$$\delta u \; = \; \phi(\mathbf{x}, u, \bar{u}, u_j, \bar{u}_j) \tag{3.3}$$

$$\delta u^j \; = \; \phi_j + \phi_u u_j + \phi_{\bar{u}} \bar{u}_j + \phi_{u_k} u_{jk} + \phi_{\bar{u}_k} \bar{u}_{jk} \tag{3.4}$$

and $\delta u^{jk} = D_k \delta u^j$ where, e.g., $D_k(\phi_u u_j)$ is

$$D_k(\phi_u u_j) = \phi_{ku} u_j + \phi_{uu} u_j u_k + \phi_{u\bar{u}} u_j \bar{u}_k + \phi_{uu_l} u_j u_{kl} + \phi_{u\bar{u}_l} u_j \bar{u}_{kl} + \phi_u u_{jk} \tag{3.5}$$

(we do not display the full expression because of its length), etc. For the details of the used formalism we refer to the book ([15]).

Now, a symmetry of a PDE $F(u, u_j, u_{jk}, \ldots) = 0$ (of n-th order, say) is a solution of the equation $\mathrm{pr}^{(n)} v(F) = 0$ which holds on-shell, i.e., when the original PDE is used together with its prolongations (PDEs that follow from $F = 0$ by applying total derivatives). We allow for a dependence of ϕ on $\mathbf{x}, u, \bar{u}, u_j, \bar{u}_j$, therefore the resulting solutions will contain the geometric target space symmetries $\phi = f(u, \bar{u})$, the geometric base space symmetries ("space-time symmetries") which in evolutionary form are given by $\phi = A^j(\mathbf{x}) u_j$ (with real A^j depending on the base space variables only) as well as "generalized symmetries" (where ϕ depends on $u_j, \bar{u}_j$ in a more general fashion).

After these preparatory remarks let us do the explicit calculation for the static eikonal equation in $d = 3$ dimensions first,

$$u_j u_j = 0 \,, \qquad j = 1, 2, 3 \tag{3.6}$$

(which generalizes immediately to the cases $d > 3$ as we shall see). The action of $\mathrm{pr}^{(1)} v$ leads to

$$(\phi_j + \phi_u u_j + \phi_{\bar{u}} \bar{u}_j + \phi_{u_k} u_{jk} + \phi_{\bar{u}_k} \bar{u}_{jk}) u_j = 0 \tag{3.7}$$

which, with the help of (3.6) and its first prolongation

$$u_j u_{jk} = 0 \tag{3.8}$$

gives

$$\phi_j u_j + \phi_{\bar{u}} u_j \bar{u}_j + \phi_{\bar{u}_k} \bar{u}_{jk} u_j = 0\,. \tag{3.9}$$

First we observe that there are no conditions on the u and u_j dependence of ϕ, therefore ϕ may be an arbitrary function of u and u_j. Next, we use that by assumption nothing may depend on $\bar{u}_{jk}$, therefore the third term must vanish separately, $\phi_{\bar{u}_k} \bar{u}_{jk} u_j = 0$. This requires either $\phi_{\bar{u}_j} = 0$ or $\phi_{\bar{u}_j} \sim \bar{u}_j \Rightarrow \phi = \phi(\bar{u}_j \bar{u}_j)$. But this argument vanishes on-shell, therefore we may conclude that $\phi_{\bar{u}_j} = 0$ without loss of generality. This implies, in turn, that the second term in (3.9) must vanish separately, because nothing may depend on $\bar{u}_j$, and , therefore, $\phi_{\bar{u}} = 0$. We are thus left with

$$\phi = \phi(\mathbf{x}, u, u_k)\,, \qquad \phi_j u_j = 0\,. \tag{3.10}$$

This does *not* imply that ϕ cannot depend on $\mathbf{x}$, but it does imply that ϕ cannot contain a term which solely depends on $\mathbf{x}$, i.e. a $\phi = f(\mathbf{x}) + \ldots$ is not permitted. An allowed term, which will lead to the base space symmetries, is $\phi = A^j(\mathbf{x}) u_j$ which has to obey

$$\phi_k u_k \equiv A^j{}_k u_j u_k = 0 \tag{3.11}$$

with the solutions

$$A^j = \text{const.} \qquad \text{or} \qquad A^j{}_k = -A^k{}_j \qquad \text{or} \qquad A^j{}_k \sim \delta^j_k \tag{3.12}$$

providing thereby the generators of the conformal group in $d = 3$ euclidean space. So we find the conformal group as the base space symmetry group and the transformations $\phi = f(u)$ for an arbitrary function f for the geometric target space symmetries.

Obviously there are many more symmetries besides the geometric symmetries (i.e., base space and geometric target space symmetries). Here we want to describe them briefly, because they are of some independent interest, although we shall not need the corresponding results for our purposes. First of all, obviously nothing depended on the fact that $d = 3$, therefore all the above results are equally true for higher dimensions $d > 3$ (with the conformal base space transformations in d dimensions, of course). Secondly, in addition to the base space symmetries (3.12) and the geometric target space symmetries $\phi = f(u)$ there are many more symmetries present for the eikonal equation. Among these are generalized symmetries $\phi = f(u_j)$ for arbitrary f, or non-projectable base space symmetries $\phi = A^j(\mathbf{x}, u) u_j$, i.e., infinitesimal conformal transformations on base space where the parameters of these transformations are arbitrary functions of u rather than constants. Even the dependence on $\mathbf{x}$ is more general than just the conformal transformations (3.12). Concretely, although each $\phi = f(A^j u_j)$ with A^j an infinitesimal conformal transformation is a symmetry, this is not the most general allowed $\mathbf{x}$ dependence. For instance, for a ϕ which is quadratic in the u_j, that is

$$\phi = B^{jk}(\mathbf{x}) u_j u_k\,, \tag{3.13}$$

the solution $B^{jk} = A^j A^k$ is not the most general solution to (3.11). The most general solution still depends on a finite number of parameters, but more than the 10 of the conformal transformations, providing thereby a kind of generalization of the conformal group in the context of generalized symmetries. Analogous results hold for higher powers of the u_j, i.e., for ϕ of the form $\phi = B^{j_1 \dots j_n}(\mathbf{x}) u_{j_1} \dots u_{j_n}$.

Next, we want to discuss the symmetries of the eikonal equation in $d = 2$ dimensions. Here it is useful to introduce the complex coordinate $z = x^1 + ix^2$, in terms of which the eikonal equation in $d = 2$ reads

$$u_z u_{\bar{z}} = 0 \tag{3.14}$$

and the first prolongation of the vector $v = \delta u \partial_u +$ c.c., where δu is given by $\delta u = \phi(z, \bar{z}, u, \bar{u}, u_z, u_{\bar{z}}, \bar{u}_z, \bar{u}_{\bar{z}})$, which has the first prolongation

$$\delta u^z = D_z \phi = \phi_z + \phi_u u_z + \phi_{\bar{u}} \bar{u}_z + \phi_{u_z} u_{zz} + \phi_{u_{\bar{z}}} u_{z\bar{z}} + +\phi_{\bar{u}_z} \bar{u}_{zz} + \phi_{\bar{u}_{\bar{z}}} \bar{u}_{z\bar{z}} \tag{3.15}$$

and a similar expression for $\delta u^{\bar{z}}$. The action of $\mathrm{pr}^{(1)} v$ leads to

$$\delta u^z u_{\bar{z}} + \delta u^{\bar{z}} u_z = 0 \tag{3.16}$$

and, upon using the eikonal equation and its two first prolongations, to

$$u_{\bar{z}} \phi_z + u_z \phi_{\bar{z}} + \phi_{\bar{u}} (\bar{u}_z u_{\bar{z}} + \bar{u}_{\bar{z}} u_z) + \phi_{\bar{u}_z} (\bar{u}_{zz} u_{\bar{z}} + \bar{u}_{z\bar{z}} u_z) + \phi_{\bar{u}_{\bar{z}}} (\bar{u}_{z\bar{z}} u_{\bar{z}} + \bar{u}_{\bar{z}\bar{z}} u_z) = 0\,. \tag{3.17}$$

This requires $\phi_{\bar{u}} = \phi_{\bar{u}_z} = \phi_{\bar{u}_{\bar{z}}} = 0$ and leads to

$$\phi_z u_{\bar{z}} + \phi_{\bar{z}} u_z = 0 \tag{3.18}$$

which has the general solution

$$\phi = F[u, u_z g(z, u), u_{\bar{z}} h(\bar{z}, u)] \tag{3.19}$$

where $F[\cdot,\cdot,\cdot]$, $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ are arbitrary functions of their arguments. Again, the geometric target space transformations $\phi = f(u)$ and the base space transformations $\phi = g(z) u_z$ (conformal transformations in $d = 2$) may be found among the symmetry transformations.

4. Symmetries of the Baby Skyrme model

The equation of motion for the Baby Skyrme model with lagrangian (1.1) in $d = 2$ dimensions is

$$(1 + u\bar{u}) \partial^\mu \partial_\mu u - 2\bar{u} (\partial^\mu u)(\partial_\mu u) = 0 \tag{4.1}$$

and the canonical momentum π for the field u is ($\dot{u} \equiv \partial_0 u$)

$$\pi = (1 + u\bar{u})^{-2} \dot{\bar{u}} \tag{4.2}$$

leading to the generators for area-preserving diffeomorphisms Q^G on the target S^2 and corresponding Noether currents J^G_μ

$$Q^G = i \int d^2\mathbf{x} (\dot{u} G_u - \dot{\bar{u}} G_{\bar{u}})\,, \qquad J^G_\mu = i(u_\mu G_u - \bar{u}_\mu G_{\bar{u}})\,. \tag{4.3}$$

Their divergence is, with the help of the equations of motion,

$$\partial^\mu J_\mu^G = iu^\mu u_\mu \left(\frac{2\bar{u}}{1+u\bar{u}} G_u + G_{uu}\right) - i\bar{u}^\mu \bar{u}_\mu \left(\frac{2u}{1+u\bar{u}} G_{\bar{u}} + G_{\bar{u}\bar{u}}\right) \tag{4.4}$$

and, as said, the currents J_μ^G are not conserved (for general G) for the Baby Skyrme model (4.1) but are conserved for the submodel defined by the eikonal equation as an additional condition.

For static, solitonic fields the field equation for the Baby Skyrme model is

$$(1+u\bar{u})u_{jj} - 2\bar{u}u_j u_j = 0 \tag{4.5}$$

and for its submodel they are

$$u_{jj} = 0 \qquad \text{and} \qquad u_j u_j = 0 \tag{4.6}$$

or $(z = x^1 + ix^2)$

$$u_{z\bar{z}} = 0 \qquad \text{and} \qquad u_z u_{\bar{z}} = 0\,. \tag{4.7}$$

(Remark: The eikonal equation $u_z u_{\bar{z}} = 0$ is closely related to the Cauchy-Riemann equations, but it is slightly more general, being the product of a holomorphic and an anti-holomorphic derivative and, therefore, also non-linear. As a consequence, the resulting symmetries are slightly different from the symmetries of the Cauchy-Riemann equations, as well.)

(Remark: as far as solitons (static solutions of finite energy) are concerned, there is not much difference between the Baby Skyrme model and its integrable submodel. Each soliton solution of the Baby Skyrme model is a rational function of the variable z only, $u = R(z)$ (or of $\bar{z}$ only), and is, therefore, a solution of the Cauchy-Riemann equations (or of its anti-holomorphic counterpart). As a consequence, it is also a solution of the submodel, as was observed, e.g. in [17].)

Now we want to discuss the symmetries both of the full static equation (4.5) and of its submodel (4.7). For the full model (4.5) the result of a long but fairly straight-forward calculation (using the prolongations (3.3)–(3.5)) is that the symmetry group is a direct product of base space and geometric target space transformations, $\phi = M(u) + A^j(\mathbf{x})u_j$, where M just generates the modular transformations,

$$M = c^a M^a\,, \qquad M^1 = \frac{1}{2}(1-u^2)\,, \qquad M^2 = \frac{1}{2}(1+u^2)\,, \qquad M^3 = iu \tag{4.8}$$

where the c^a are three real constants, and A^j are the infinitesimal conformal transformations in $d = 2$, i.e., $A^1 + iA^2 = f(z)$ where $z = x^1 + ix^2$ and f is an arbitrary function of its argument. As this result is well-known and hardly surprising, we do not show the details of the calculation.

For the symmetry calculations of the submodel we may use the result (3.19) from the eikonal equation that $\phi = F[u, u_z g(z,u), u_{\bar{z}} h(\bar{z}, u)]$. Further we need the second prolongation

$$\delta u^{z\bar{z}} = D_{\bar{z}} D_z \phi = D_{\bar{z}}(\phi_z + \phi_u u_z + \phi_{u_z} u_{zz} + \phi_{u_{\bar{z}}} u_{z\bar{z}}) = \cdots \tag{4.9}$$

where we do not display the lengthy expression explicitly. The condition $\mathrm{pr}^{(2)}v(u_{z\bar{z}}) = 0$ just means $\delta u^{z\bar{z}} = 0$ which, using equations (4.7) and their first prolongations, leads to

$$\phi_{z\bar{z}} + \phi_{zu}u_{\bar{z}} + \phi_{\bar{z}u}u_z + \phi_{zu_{\bar{z}}}u_{\bar{z}\bar{z}} + \phi_{\bar{z}u_z}u_{zz} + \phi_{u_z u_{\bar{z}}}u_{zz}u_{\bar{z}\bar{z}} = 0 \tag{4.10}$$

where now each coefficient has to vanish separately. The condition $\phi_{u_z u_{\bar{z}}} = 0$ implies that ϕ is a direct sum of two types of functions,

$$\phi = F^1[u, u_z f(z,u)] + F^2[u, u_{\bar{z}} g(\bar{z},u)] \tag{4.11}$$

and we assume now that $\phi = F^1[u, u_z f(z)]$ (the second case may be treated analogously). All conditions are then fulfilled identically except for $\phi_{zu} = 0$ which leads to

$$F^1{}_{ab}u_z f_z + F^1{}_{bb}u_z^2 f_u f_z + F^1{}_b u_z f_{zu} = 0 \tag{4.12}$$

(here $F^1 \equiv F^1[a,b]$, $a \equiv u$, $b \equiv u_z f(z,u)$) and is solved either by

$$f_z \equiv 0 \quad \Rightarrow \quad \phi = G(u, u_z) \tag{4.13}$$

or by

$$f_u = 0 \quad \wedge \quad F^1{}_a = 0 \qquad \Rightarrow \qquad \phi = F[u_z f(z)]\,. \tag{4.14}$$

Of course, the coordinate transformations $\phi = u_z f(z)$ and the geometric target space transformations $\phi = G(u)$ are among the symmetries we found. Furthermore, we see that the target space symmetries are not related to the area-preserving diffeomorphisms, which would require $\phi = i(1+u\bar{u})^2 G_{\bar{u}}$ for an arbitrary real $G(u,\bar{u})$.

All in all, we find that the submodel has the generalized symmetries given by

$$\phi = F[u_z f(z)] + G(u, u_z) + \tilde{F}[u_{\bar{z}} g(\bar{z})] + \tilde{G}(u, u_{\bar{z}}) \tag{4.15}$$

and, therefore, has certainly more symmetries than the full Baby Skyrme model.

(Remark: Solutions to the submodel $u_k u_k = 0 \wedge u_{kk} = 0$ solve, in fact, a much larger class of models. Indeed, if we start with the lagrangian density

$$\mathcal{L} = f(u,\bar{u})\partial^\mu u \partial_\mu \bar{u} \tag{4.16}$$

for a general real function f of its arguments, then the resulting Euler-Lagrange equation is

$$f\partial^\mu\partial_\mu u + f_u \partial^\mu u \partial_\mu u = 0 \tag{4.17}$$

and, in the static case, is solved by any solution of the submodel. The same reasoning also shows that the pair of equations $u_k u_k = 0 \wedge u_{kk} = 0$ cannot have non-pathological solutions in more than two dimensions, because they would solve the above equation of motion and, thereby, violate Derrick's theorem (here pathological means that any formal solution of the equation pair must lead to an infinite energy, whatever the function $f(u,\bar{u})$ in the lagrangian is).)

JHEP01(2005)004

5. Symmetries of the Nicole model

The Nicole model (1.6) leads to the equation of motion

$$\frac{1}{2}(u^\lambda \bar{u}_\lambda)_\mu u^\mu + u^\lambda \bar{u}_\lambda u^\mu{}_\mu - \frac{u^\lambda \bar{u}_\lambda}{1+u\bar{u}}(u^\mu \bar{u}_\mu u + 3u^\mu u_\mu \bar{u}) = 0 \tag{5.1}$$

and to the canonical momentum

$$\pi = \frac{3}{2}(1+u\bar{u})^{-3}(u^\mu \bar{u}_\mu)^{\frac{1}{2}}\dot{\bar{u}}\,. \tag{5.2}$$

The Noether currents for the area-preserving diffeomorphisms are

$$J^G{}_\mu = i(1+u\bar{u})^{-1}(u^\lambda \bar{u}_\lambda)^{\frac{1}{2}}(u_\mu G_u - \bar{u}_\mu G_{\bar{u}}) \tag{5.3}$$

and their divergence may be computed with the help of the equation of motion to be

$$\partial^\mu J^G{}_\mu = i\frac{(u^\lambda \bar{u}_\lambda)^{\frac{1}{2}}}{1+u\bar{u}}\left(u^\mu u_\mu \left(\frac{2\bar{u}}{1+u\bar{u}}G_u + G_{uu}\right) - \bar{u}^\mu \bar{u}_\mu \left(\frac{2u}{1+u\bar{u}}G_{\bar{u}} + G_{\bar{u}\bar{u}}\right)\right). \tag{5.4}$$

Again, the currents are not conserved for general G for the full Nicole model, but they are conserved for the submodel when u obeys the eikonal equation, as well.

The static field equation for the full Nicole model is

$$\frac{1}{2}(u_{jk}\bar{u}_j u_k + \bar{u}_{jk} u_j u_k) + u_j \bar{u}_j u_{kk} - \frac{u_j \bar{u}_j}{1+u\bar{u}}(u_k \bar{u}_k u + 3u_k u_k \bar{u}) = 0 \tag{5.5}$$

and for the submodel the equations are

$$\frac{1}{2}\bar{u}_{jk} u_j u_k + u_j \bar{u}_j u_{kk} - \frac{(u_j \bar{u}_j)^2 u}{1+u\bar{u}} = 0 \qquad \wedge \qquad u_j u_j = 0\,. \tag{5.6}$$

(Remark: as already mentioned, the known results on this model are rather scarce. The only known analytic solution of the static equation (5.5) is the simplest Hopf map [11]

$$u = \frac{x^1 + ix^2}{2x^3 - i(1-r^2)} \tag{5.7}$$

(here $r^2 \equiv x^j x^j$). However, it is known that the simplest Hopf map also solves the eikonal equation ([16]), and, consequently, the submodel (5.6) ([17]). In spite of the scarce results we know, therefore, that the solution space of the submodel is non-empty.)

Now we want to calculate the symmetries of the static equations both for the full Nicole model and for its submodel. For the full model a long calculation, similarly to the case of the Baby Skyrme model, shows that the symmetry group is again a direct product of base space and geometric target space transformations, $\phi = M(u) + A^j(\mathbf{x})u_j$, where, again, M generates the modular transformations, see (4.8), and A^j are the generators of the infinitesimal conformal transformations in $d = 3$, see (3.12).

For the calculation of the symmetries of the submodel we want to briefly sketch the most important steps. We know from the symmetries of the eikonal equation that ϕ is of the form $\phi = \phi(\mathbf{x}, u, u_j)$, see (3.10). The action of the second prolongation of the symmetry-generating vector field on the submodel equation leads to

$$\frac{1}{2}\bar{\phi}^{jk} u_j u_k + \bar{u}_{jk}\phi^j u_k + \phi^{jj} u_k \bar{u}_k + u_{jj}(\phi^k \bar{u}_k + \bar{\phi}^k u_k) - \\ -2u_j\bar{u}_j(\phi^k \bar{u}_k + \bar{\phi}^k u_k)\frac{u}{1+u\bar{u}} - (u_j\bar{u}_j)^2 \frac{\phi - u^2\bar{\phi}}{(1+u\bar{u})^2} = 0\,. \tag{5.8}$$

There are two terms in this expression which contain third derivatives of u, namely the first term, which contains $\bar{\phi}_{\bar{u}_l}\bar{u}_{jkl}u_j u_l$ and the third term, which contains $u_j\bar{u}_j\phi_{u_l}u_{kkl}$. With the help of the first prolongation of the field equation the first term may be re-expressed like $\bar{\phi}_{\bar{u}_l}(-u_j\bar{u}_j u_{kkl} + \cdots)$, where the remainder contains only second derivatives. Cancellation of the two terms with third derivatives now requires $\bar{\phi}_{\bar{u}_l} = \phi_{u_l}$ which implies $\phi = \psi(\mathbf{x}, u) + A^k(\mathbf{x})u_k$. From the symmetry results of the eikonal equation we may further conclude that $\psi = \psi(u)$ and that the A^k are just the generators of the conformal transformations in $d = 3$ dimensions. It remains to determine ψ. The coefficients of the terms with second derivatives give no further conditions (i.e., they either cancel mutually or vanish identically for a ϕ of the above form), and the terms containing only first derivatives give just one further condition,

$$(u_j\bar{u}_j)^2 \left(\frac{1}{2}\bar{\psi}_{\bar{u}\bar{u}} - \frac{u\bar{\psi}_{\bar{u}}}{1+u\bar{u}} - \frac{\psi - u^2\bar{\psi}}{(1+u\bar{u})^2}\right) = 0 \tag{5.9}$$

which is solved by the modular transformations, $\psi(u) = M(u)$. Therefore, the submodel does not have more symmetries than the full Nicole model, in spite of the infinitely many conserved currents of the former.

6. Symmetries of the Faddeev-Niemi model

The Faddeev-Niemi model (1.3) leads to the equation of motion

$$(1+u\bar{u})\partial^\mu \left(u_\mu - 2\lambda(1+u\bar{u})^{-2}(\bar{u}^\nu u_\nu u_\mu - u^\nu u_\nu \bar{u}_\mu)\right) - 2\bar{u}u^\mu u_\mu = 0 \tag{6.1}$$

and to the canonical momentum

$$\pi = \frac{\dot{\bar{u}}}{(1+u\bar{u})^2} - 2\lambda \frac{u^\mu \bar{u}_\mu \dot{\bar{u}} - \bar{u}^\mu \bar{u}_\mu \dot{u}}{(1+u\bar{u})^4}\,. \tag{6.2}$$

The Noether currents for the area-preserving diffeomorphisms are

$$J^G{}_\mu = i\left(u_\mu - \frac{2\lambda}{(1+u\bar{u})^2}(u^\nu \bar{u}_\nu u_\mu - u^\nu u_\nu \bar{u}_\mu)\right) G_u - \\ -i\left(\bar{u}_\mu - \frac{2\lambda}{(1+u\bar{u})^2}(u^\nu \bar{u}_\nu \bar{u}_\mu - \bar{u}^\nu \bar{u}_\nu u_\mu)\right) G_{\bar{u}} \tag{6.3}$$

and their divergence may be computed with the help of the equation of motion to be

$$\partial^\mu J^G{}_\mu = i\left(u^\mu u_\mu \left(\frac{2\bar{u}}{1+u\bar{u}}G_u + G_{uu}\right) - \bar{u}^\mu \bar{u}_\mu \left(\frac{2u}{1+u\bar{u}}G_{\bar{u}} + G_{\bar{u}\bar{u}}\right)\right). \tag{6.4}$$

Here, again, the currents are not conserved for general G for the full Faddeev-Niemi model, but they are conserved for the submodel where u obeys the eikonal equation.

(Remark: Observe that the above divergence does not contain a term proportional to λ. This shows that the second lagrangian $\mathcal{L}_4$ alone (remember that $\mathcal{L}_{\rm FN} = \mathcal{L}_2 - \lambda \mathcal{L}_4$, see eq. (1.3)) is invariant under area-preserving diffeomorphisms as well as arbitrary powers of this lagrangian, demonstrating the invariance of the AFZ model.)

The static field equations for the full Faddeev-Niemi model are

$$\begin{aligned}(1+u\bar{u})^3 u_{kk} - 2(1+u\bar{u})^2 \bar{u} u_k u_k - 4\lambda u \left((u_j \bar{u}_j)^2 - u_j u_j \bar{u}_k \bar{u}_k \right) + \\ +2\lambda (1+u\bar{u}) \left(\bar{u}_{jk} u_j u_k - u_{jk} u_j \bar{u}_k + \bar{u}_j u_j u_{kk} - u_j u_j \bar{u}_{kk} \right) = 0 \,,\end{aligned} \tag{6.5}$$

and for the submodel they are

$$(1+u\bar{u})^3 u_{kk} - 4\lambda u (u_j \bar{u}_j)^2 + 2\lambda (1+u\bar{u})(\bar{u}_{jk} u_j u_k + u_j \bar{u}_j u_{kk}) = 0 \tag{6.6}$$

together with the eikonal equation.

(Remark: results on static solutions of the Faddeev-Niemi model are again quite scarce. Only numerical results on the simplest solitons with low topological index exist ([3]–[5]). Besides, in this case — contrary to the Nicole model — it is not known whether the solution space of the submodel is empty or non-empty. Recently a class of exact solutions (both static and non-static) has been constructed in [18], but the resulting solutions are not "simple" (e.g., they do not have obvious symmetries), and it is not known whether they correspond, in the static cases, to true minima of the energy within a given topological sector, or whether they are critical points of another type.)

As the symmetry calculations are quite lengthy, we just quote the results here. It turns out that again the submodel does not have more symmetry than the full Faddeev-Niemi model and that the symmetries are just the expected ones. The symmetry-generating vector field $v = \phi \partial_u$ is given by $\phi = M(u) + A^k(\mathbf{x}) u_k$, where M again generates the modular transformations (4.8), whereas A^k this time has to obey

$$A^j = \text{const.} \qquad \text{or} \qquad {A^j}_k = -{A^k}_j \tag{6.7}$$

i.e., it generates translations and rotations (conformal transformations being absent because the static Faddeev-Niemi model is not scale invariant and contains the dimensionful coupling constant λ).

7. Conclusions

We have thoroughly analysed the symmetries of submodels of some relevant solitonic relativistic theories defined in two and three dimensions on target space S^2, which have an infinite number of conserved currents. They arose in various attempts to apply a generalization of the zero curvature representation to the CP^1 (Baby Skyrme), the Faddeev and Niemi S^2 restriction of Skyrme theory and proposals to overcome the Derrick scaling with one of the terms, specially the quadratic one due to Nicole, as the quartic AFZ case has

model	∞ many conserv. laws	geometric symmetries	generalized symmetries	solutions known
Baby Skyrme	yes*	$C_2 \times \mathrm{SU}(2)$	no	yes
submodel	yes	$C_2 \times C_2$	yes	yes
Nicole model	no	$C_3 \times \mathrm{SU}(2)$	no	yes
submodel	yes	$C_3 \times \mathrm{SU}(2)$	no	yes
Faddeev-Niemi	no	$E_3 \times \mathrm{SU}(2)$	no	yes
submodel	yes	$E_3 \times \mathrm{SU}(2)$	no	no

Table 1: Symmetries and conservation laws of the three soliton models and their submodels. C_d: conformal group in d dimensions. E_d: euclidean group (translations and rotations) in d dimensions. *Due to the infinite-dimensional base space symmetries C_2.

been already extensively studied. All submodels are parametrized by a complex field and defined by the eikonal equation $(\partial u)^2 = 0$.

The *prolongation* method may be not so well-known in this physical context. We have therefore done some effort to explain it, giving detailed expressions. Furthermore, we calculated the canonical momenta and Poisson structures, which may be useful for future work on these theories. This should be the case, as well, for the analysis of the eikonal equation and area preserving diffeomorphisms. Specifically, for the eikonal equation we found a rather large symmetry, which may be of independent interest.

The general result for all cases is that the area-preserving diffeomorphisms are not symmetries of any eikonal submodel. Also, the three-dimensional submodels of Faddeev-Niemi and Nicole have no additional symmetries compared to the full theories.

The Baby Skyrme model is special, as the restriction does have an intriguing additional symmetry. This can be important as there is not much difference of the solutions of the full model and the restriction, at least for the static case. Finally we remind that the method can be easily extended to include the time dependence.

Finally, we want to present our results on the symmetries of the three models in table 1.

Acknowledgments

The authors thank L. Ferreira for helpful remarks on the paper. This research was partly supported by MCyT(Spain) and FEDER (FPA2002-01161), Incentivos from Xunta de Galicia and the EC network "EUCLID". Further, CA acknowledges support from the Austrian START award project FWF-Y-137-TEC and from the FWF project P161 05 NO 5 of N.J. Mauser.

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JHEP01(2005)004

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JHEP01(2005)004